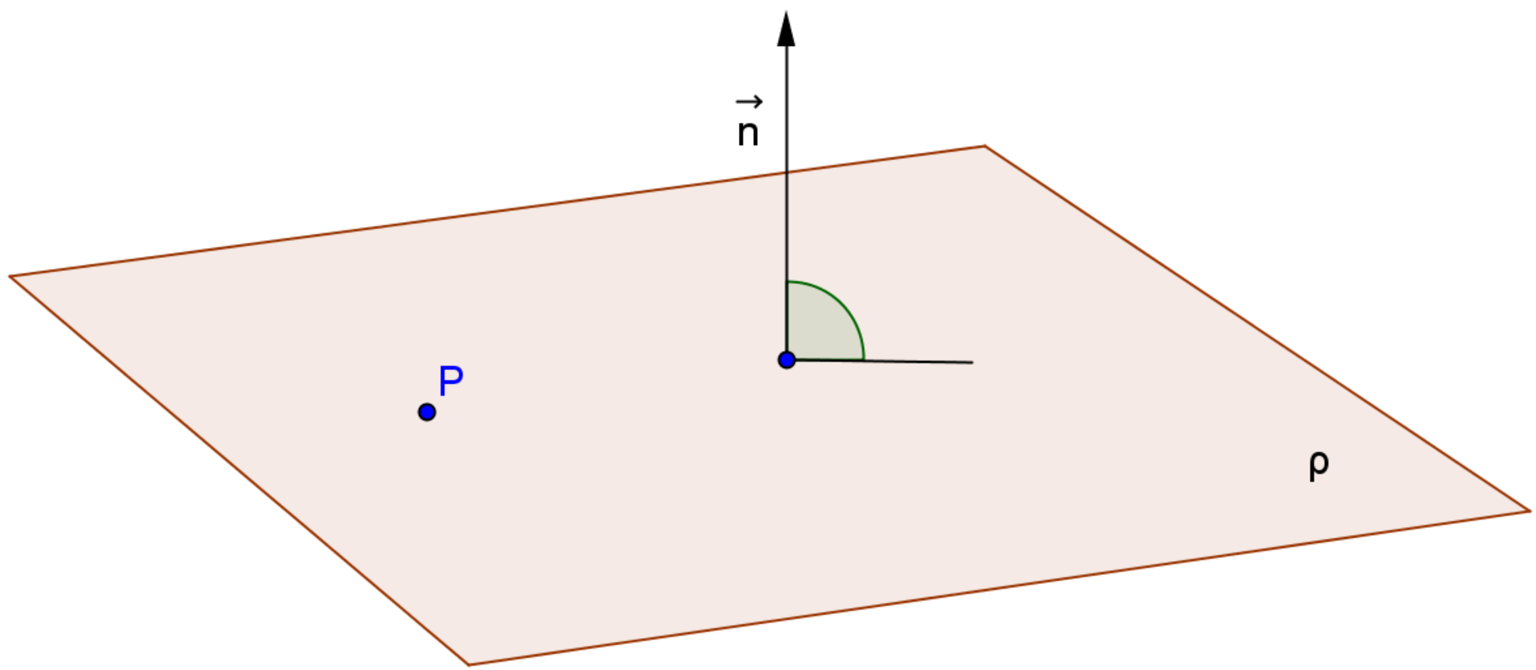
ANALYTIC GEOMETRY of the EUCLIDEAN SPACE E^3

PLANE

Plane in the space is uniquely defined by

1. three different non-collinear points
2. two intersecting lines
3. two different parallel lines
4. line and point not on this line
5. point and direction perpendicular to the plane

Plane can be uniquely determined by an arbitrary point *P* and vector $\vec{\mathbf{n}}$ perpendicular to the plane.



Any non-zero vector $\vec{\mathbf{n}}$, which is perpendicular to the plane, is called normal vector to the plane.

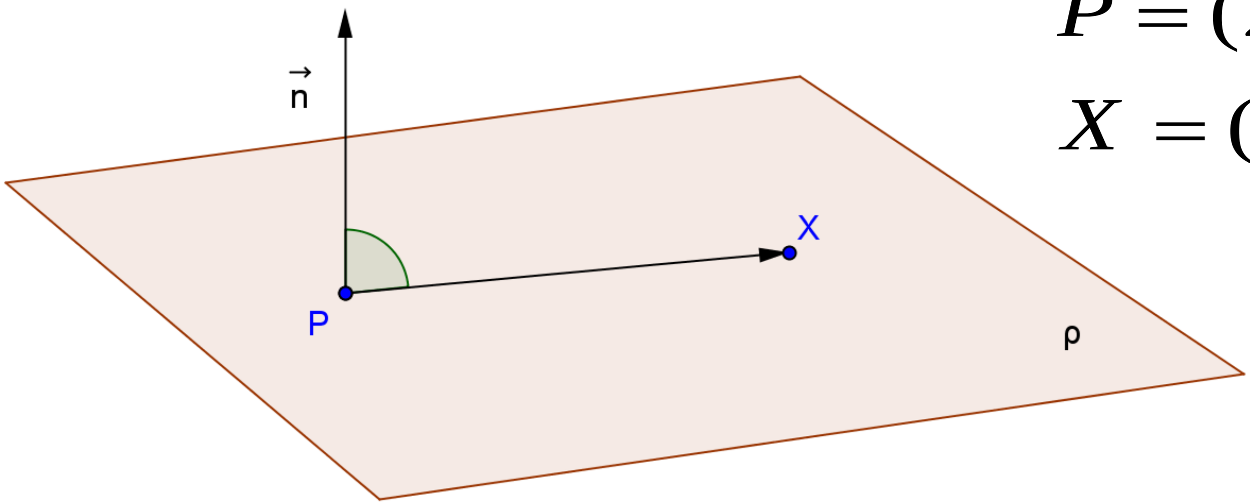
For an arbitrary point X of the plane ρ holds

$$\vec{PX} \perp \vec{\mathbf{n}} \Leftrightarrow \vec{PX} . \vec{\mathbf{n}} = 0$$

$$\vec{\mathbf{n}} = (a, b, c)$$

$$P = (x_P, y_P, z_P)$$

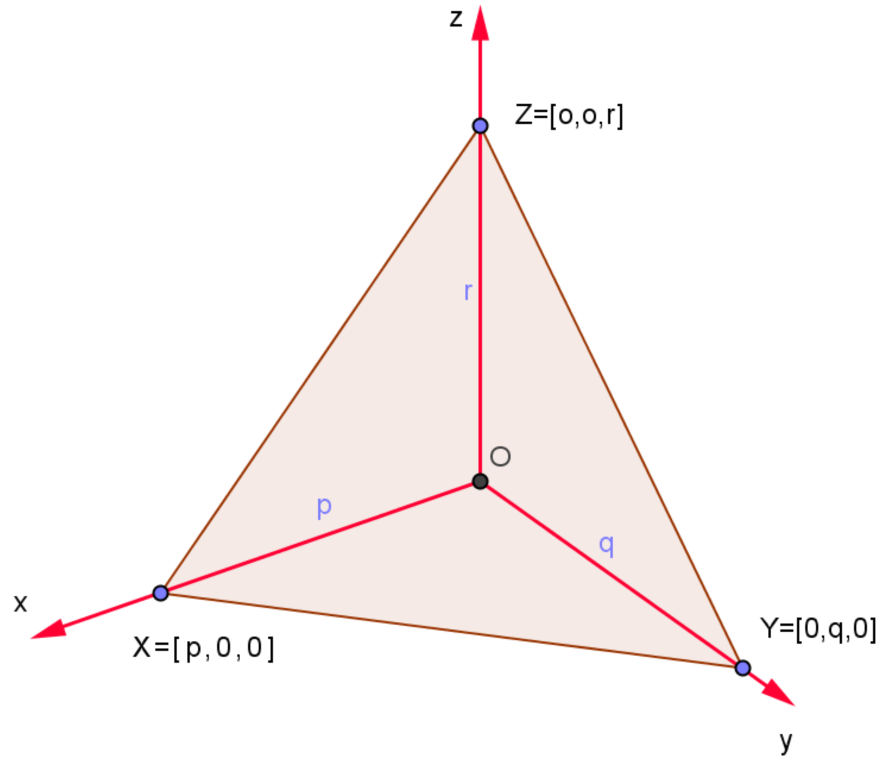
$$X = (x, y, z)$$



General equation of the plane

$$\rho : ax + by + cz + d = 0, a^2 + b^2 + c^2 \neq 0$$

$$O = [0,0,0] \in \rho \Leftrightarrow d = 0, a^2 + b^2 + c^2 \neq 0$$



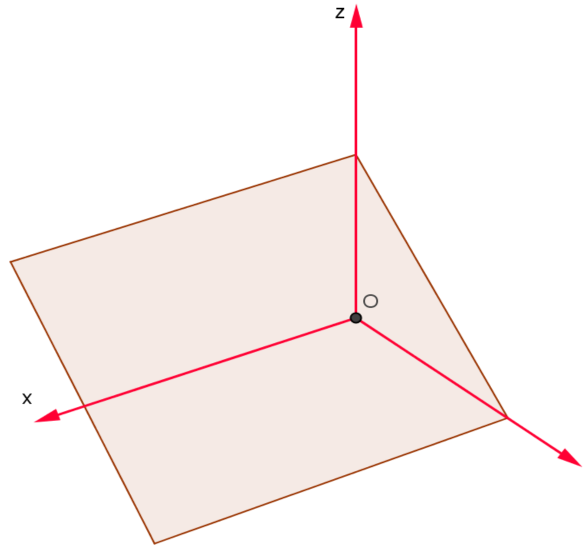
Intercept form of the equation of plane given by three points *X*, *Y*, *Z*

$$\frac{x}{p}+\frac{y}{q}+\frac{z}{r}=1, \qquad p,q,r\neq 0$$

Special positions - plane perpendicular to the coordinate plane (parallel to the coordinate axis not in this plane)

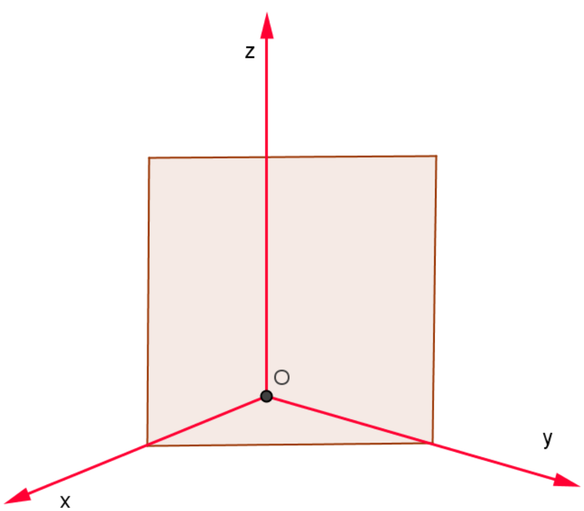
$$\rho \perp \mathbf{R}_{yz}, \rho \parallel x$$

$$by + cz + d = 0, \quad b \neq 0, c \neq 0$$



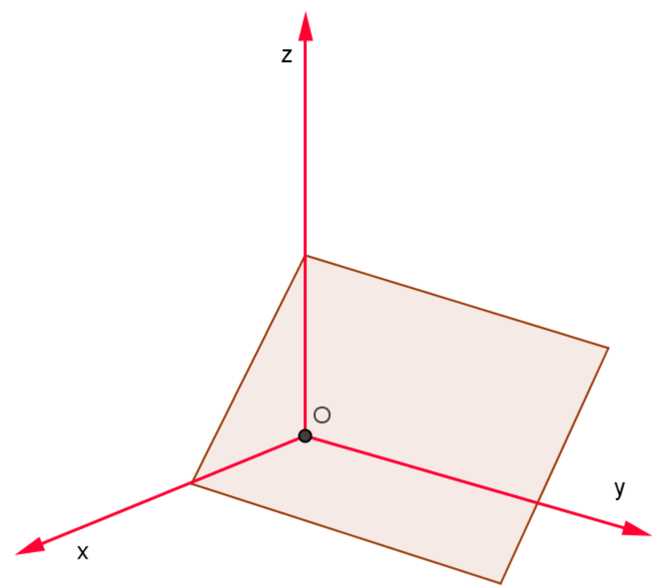
$$\rho \perp \mathbf{R}_{xy}, \rho \parallel z$$

$$ax + by + d = 0, \quad a \neq 0, b \neq 0$$



$$\rho \perp \mathbf{R}_{xz}, \rho \parallel y$$

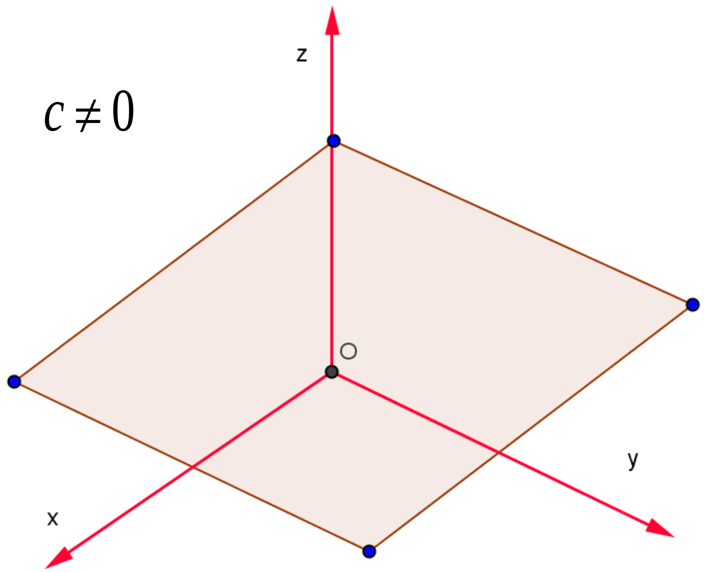
$$ax + cz + d = 0, \quad a \neq 0, c \neq 0$$



Special positions - plane parallel to the coordinate plane

$$\rho \parallel \mathbf{R}_{xy} : cz + d = 0, \quad c \neq 0$$

$$\mathbf{R}_{xy} : z = 0$$

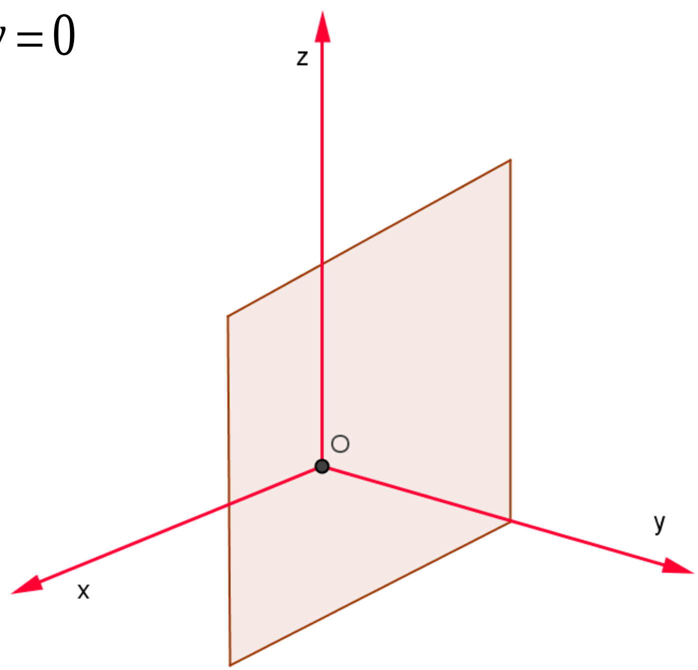


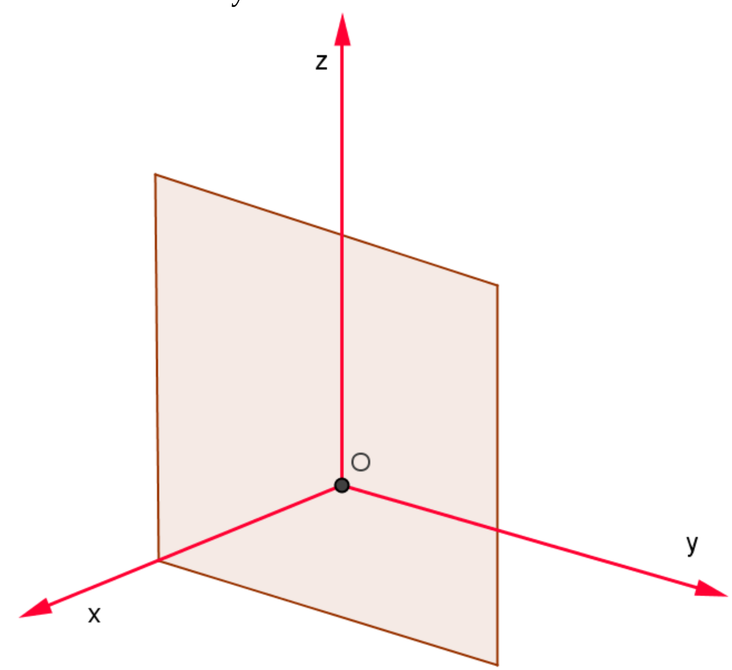
$$\rho \parallel \mathbf{R}_{yz} : ax + d = 0, \quad a \neq 0$$

$$\mathbf{R}_{yz} : x = 0$$

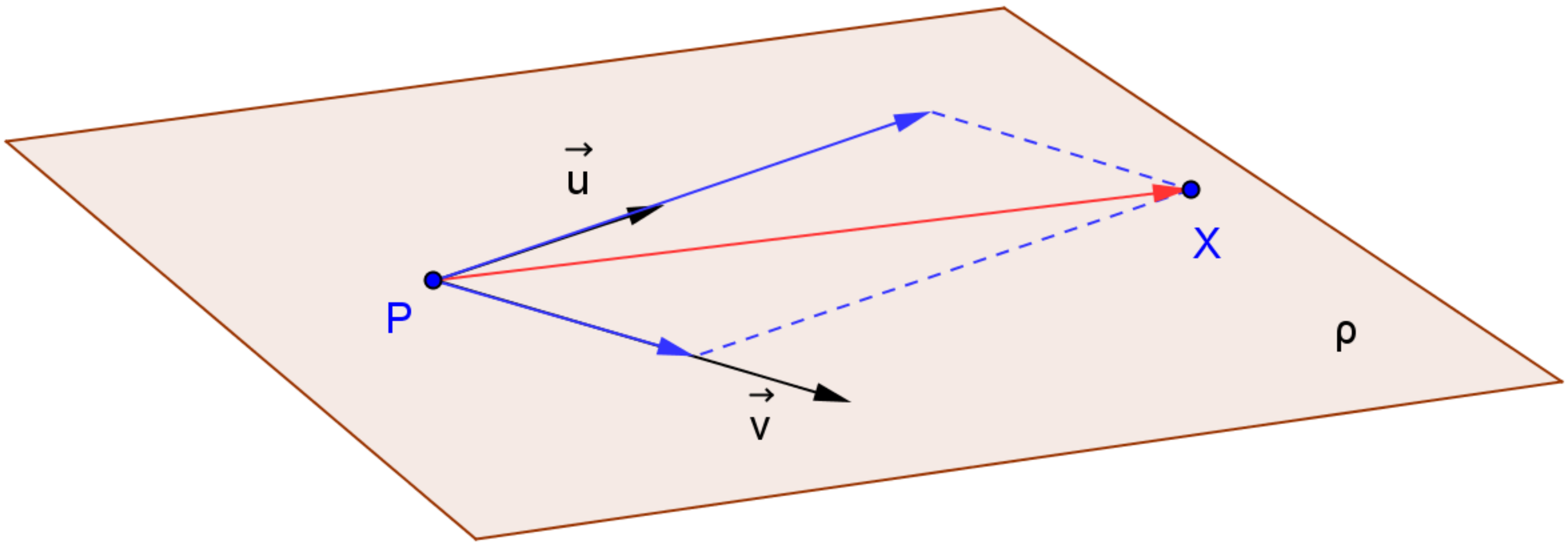
$$\rho \parallel \mathbf{R}_{xz} : by + d = 0, \quad b \neq 0$$

$$\mathbf{R}_{xz} : y = 0$$





Plane can be uniquely defined by one arbitrary point *P* and two non-collinear direction vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}$



Any point in the plane can be determined as particular linear combination of the plane direction vectors

$$\vec{PX} = t.\vec{\mathbf{u}} + s.\vec{\mathbf{v}}, \quad t, s \in R$$

Symbolic parametric equations of the plane

$$\vec{PX} = X - P = t.\vec{\mathbf{u}} + s.\vec{\mathbf{v}}, \quad t, s \in R$$

$$X = P + t.\vec{\mathbf{u}} + s.\vec{\mathbf{v}}, \quad t, s \in R$$

Parametric equations of the plane

$$P = [x_P, y_P, z_P], \vec{\mathbf{u}} = (u_1, u_2, u_3), \vec{\mathbf{v}} = (v_1, v_2, v_3)$$

$$x = x_P + t.u_1 + s.v_1$$

$$y = y_P + t.u_2 + s.v_2$$

$$z = z_P + t.u_3 + s.v_3$$

$$X = [x, y, z], t, s \in R$$

Mutual position of 2 planes

- **parallel**
  - perpendicular to the same direction of their normal vector $\vec{\mathbf{n}}$, no common points

$$\alpha : ax + by + cz + d_1 = 0,$$

$$\beta : ax + by + cz + d_2 = 0$$

$$a^2 + b^2 + c^2 \neq 0$$

$$\vec{\mathbf{n}} = (a, b, c)$$

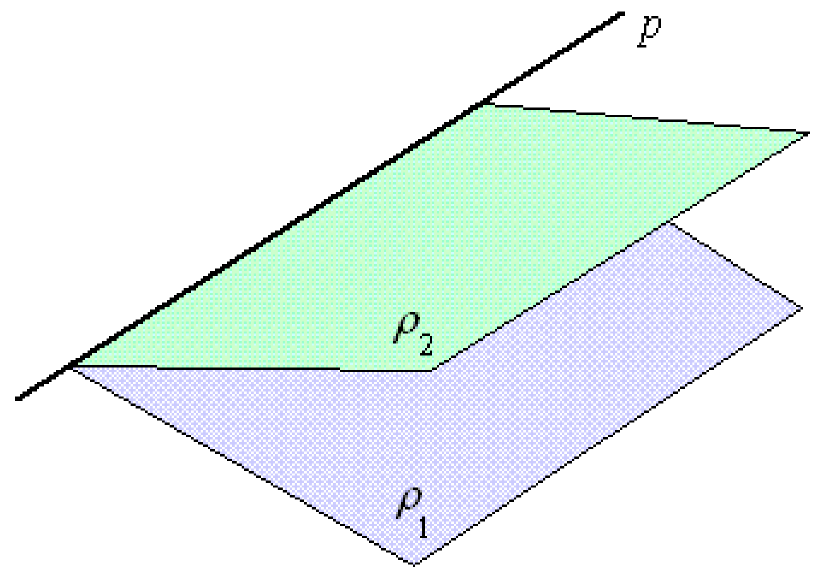
- **Intersecting**
  - 1 common line, intersection – pierce line

Line can be determined uniquely as intersection of two planes

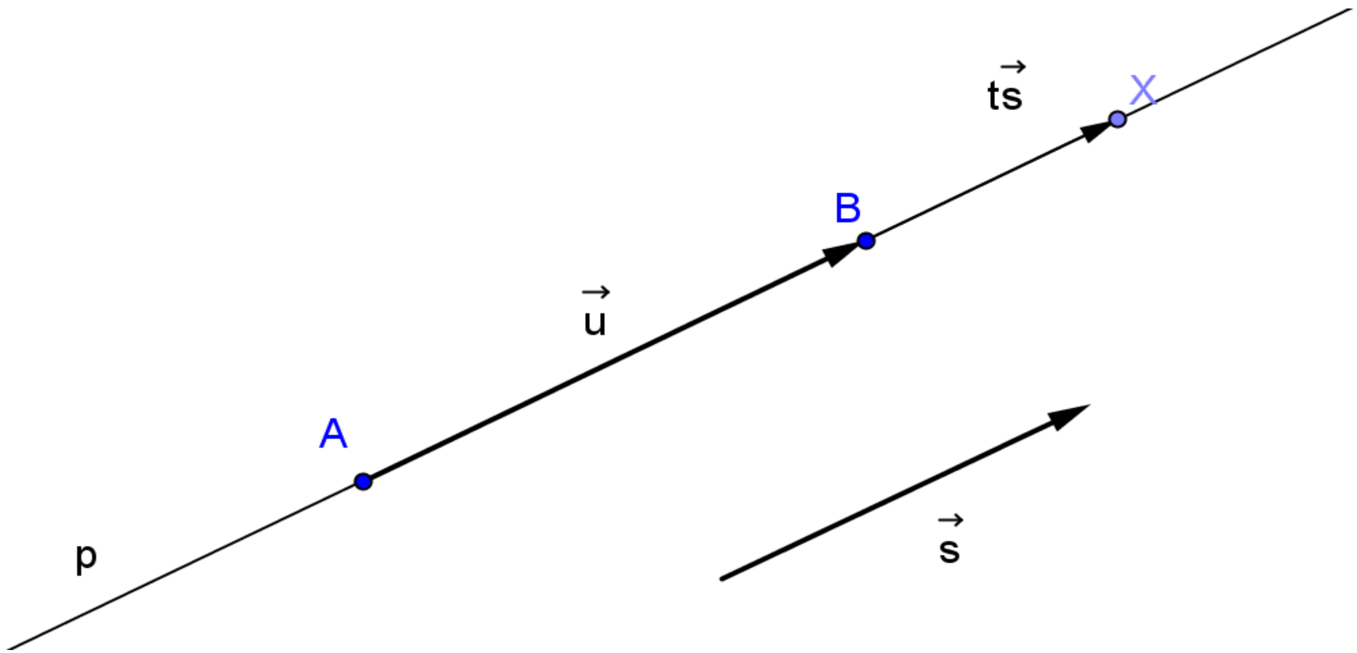
$$\rho_1 : a_1x + b_1y + c_1z + d_1 = 0, \quad a_1^2 + b_1^2 + c_1^2 \neq 0$$

$$\rho_2 : a_2x + b_2y + c_2z + d_2 = 0, \quad a_2^2 + b_2^2 + c_2^2 \neq 0$$

$$p = \rho_1 \cap \rho_2$$



Line can be uniquely determined by two different points, *p* = *AB*, or one arbitrary point *A* and direction vector $\vec{\mathbf{u}}$



Any non-zero vector parallel to the line is a line direction vector.
Any vector determined by 2 points on the line can be determined also as scalar multiple of the line direction vector

$$\vec{AX} = t.\vec{\mathbf{s}}, \quad t \in R$$

Symbolic parametric equations of the line

$$\overrightarrow{AX} = X - A = t.\vec{\mathbf{s}}, \quad t \in R$$

$$X = A + t.\vec{\mathbf{s}}, \quad t \in R$$

Parametric equations of the line

$$A = [x_A, y_A, z_A], \vec{\mathbf{s}} = (s_1, s_2, s_3)$$

$$x = x_A + t.s_1$$

$$y = y_A + t.s_2$$

$$z = z_A + t.s_3$$

$$X = [x, y, z], t \in R$$

Mutual position of 2 lines

- **parallel**
  - collinear direction vectors

$$\vec{\mathbf{s}}_2 = k.\vec{\mathbf{s}}_1, k \in R$$

$$X_1 = A + u.\vec{\mathbf{s}}_1, \quad u \in R$$

$$X_2 = B + v.\vec{\mathbf{s}}_2, \quad v \in R$$

- **intersecting**
  - one common point

$$M = A + u_M.\vec{\mathbf{s}}_1 = B + v_M.\vec{\mathbf{s}}_2, \quad u_M, v_M \in R$$

- **skew**
  - non-parallel, no common point

$$\vec{\mathbf{s}}_2 \neq k.\vec{\mathbf{s}}_1, k \in R$$

$$X_1 = A + u.\vec{\mathbf{s}}_1, \quad u \in R$$

$$X_2 = B + v.\vec{\mathbf{s}}_2, \quad v \in R$$

Mutual position of line and plane

- **parallel**
  - coplanar direction vectors

$$\vec{\mathbf{s}}_1 = k.\vec{\mathbf{s}}_2 + l.\vec{\mathbf{s}}_3, k, l \in R$$

$$X_1 = A + t.\vec{\mathbf{s}}_1, \quad t \in R$$

$$X_2 = B + u.\vec{\mathbf{s}}_2 + v.\vec{\mathbf{s}}_3, \quad u, v \in R$$

  - perpendicular line direction vector $\mathbf{s}_1$ and plane normal vector $\mathbf{n} = \mathbf{s}_2 \times \mathbf{s}_3$, $\quad \mathbf{s}_1.\mathbf{n} = 0$

- **intersecting**
  - one common point

$$M = A + t_M.\vec{\mathbf{s}}_1 = B + u_M.\vec{\mathbf{s}}_2 + v_M.\vec{\mathbf{s}}_3$$

$$t_M, u_M, v_M \in R$$

Measuring distances

- Distance of two points
  - Distance of point $A = [x_0, y_0, z_0]$ and plane $ax + by + cz + d = 0$

$$d(A, \rho) = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

  - Distance of point and line
  - Distance of parallel lines
  - Distance of parallel planes $ax + by + cz + d_1 = 0$

$$ax + by + cz + d_2 = 0$$

$$d(\rho_1, \rho_2) = \frac{|d_1 - d_2|}{|\vec{\mathbf{n}}|}, \vec{\mathbf{n}} = (a, b, c)$$

Measuring angles

- Angle of two lines $A\vec{\mathbf{u}}, B\vec{\mathbf{v}}, \quad \cos\varphi = \dfrac{\vec{\mathbf{u}}.\vec{\mathbf{v}}}{|\vec{\mathbf{u}}|.|\vec{\mathbf{v}}|}$
  - Angle of line $A\vec{\mathbf{u}}$
    and plane $ax + by + cz + d = 0$

$$\cos(90^\circ - \varphi) = \frac{\vec{\mathbf{u}}.\vec{\mathbf{n}}}{|\vec{\mathbf{u}}|.|\vec{\mathbf{n}}|}, \vec{\mathbf{n}} = (a,b,c)$$

  - Angle of two planes $a_1x + b_1y + c_1z + d_1 = 0$
    $a_2x + b_2y + c_2z + d_2 = 0$

$$\cos(\varphi) = \frac{\vec{\mathbf{n}}_1.\vec{\mathbf{n}}_2}{|\vec{\mathbf{n}}_1|.|\vec{\mathbf{n}}_2|}, \vec{\mathbf{n}}_1 = (a_1,b_1,c_1), \vec{\mathbf{n}}_2 = (a_2,b_2,c_2)$$